Summer Bridge Course: Analysis

Dr. Kahn

Infimum/Supremum

Upper bound: $A \subseteq \mathbb{R}$ is bounded above if $\exists U \in \mathbb{R}$ such that $a \leq U, \forall a \in A$.
U is called the upper bound of A.

Lower bound: $A \subseteq \mathbb{R}$ is bounded below if $\exists L \in \mathbb{R}$ such that $L \leq a, \forall a \in A$.
L is called the lower bound of A.

Bounded: A is bounded if A is bounded both above and below $(L \leq a \leq U, \forall a \in A)$

Supremum: $E \subseteq \mathbb{R}$, $E \neq \emptyset$. $\alpha = \sup(E)$ if

1. $x \leq \alpha, \forall x \in E$
2. If $\gamma \in \mathbb{R}$ and $\gamma < \alpha$, then γ is not an upperbound of E.

Infimum: $E \subseteq \mathbb{R}$, $E \neq \emptyset$. $\beta = \inf(E)$ if

1. $\beta \leq x, \forall x \in E$
2. If $\gamma \in \mathbb{R}$ and $\gamma > \beta$, then γ is not a lower bound of E.

Completeness Property:

1. $E \subseteq \mathbb{R}, E \neq \emptyset$, and E is bounded above, then there exists $\alpha \in \mathbb{R}$ such that $\alpha = \sup(E)$. (Note: α may or may not be in E.)
2. $E \subseteq \mathbb{R}, E \neq \emptyset$, and E is bounded below, then there exists $\beta \in \mathbb{R}$ such that $\beta = \inf(E)$. (Note: β may or may not be in E.)

Archimedian Property: If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $nx > y$.

Density of $\mathbb{Q}$: If $x, y \in \mathbb{R}$ and $x < y$, then there always exists an $r \in \mathbb{Q}$ such that $x < r < y$.

Sequences in $\mathbb{R}$

Convergence: $\{x_n\}$ converges to $x \in \mathbb{R}$ if $\forall \epsilon > 0, \exists n_0(\epsilon)$ such that $|s_n - x| < \epsilon, \forall n > n_0$.

Proof Outline: To show $\{x_n\}$ converges to x:

1. Do scratch work to find $|x_n - x| <$ (term involving n) $< \epsilon$.
2. Choose n_0 based off your scratch work.
3. Write out proof and include scratch work.

Diverges: $\{x_n\}$ diverges to ∞ if $\forall M > 0, \exists n_0(M)$ such that $x_n > M, \forall n > n_0$.

Triangle Inequality:

- $|x + y| \leq |x| + |y|$
- $||x| - |y|| \leq |x - y|$

Theorem: If $\{x_n\}$ is a convergent sequence, then $\{x_n\}$ is bounded.

Theorem: Let $a_n \to a$ and $b_n \to b$.

- $a_n \pm b_n \to a \pm b$
- $a_n b_n \to ab$
- $\frac{b_n}{a_n} \to \frac{b}{a}$ as long as $a_n \neq 0, a \neq 0, \forall n \in \mathbb{N}$
- $a_n + c \to a + c$, $c \in \mathbb{R}$
- $ca_n \to ca$, $c \in \mathbb{R}$

Theorem: If $a_n \to 0$ and b_n is bounded, then $a_n b_n \to 0$.

Squeeze Lemma: Let a_n, b_n, c_n be sequences of real numbers such that $a_n \leq b_n \leq c_n, \forall n \geq n_0$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

Sequences in $\mathbb{R}$ continued:

Monotone Increasing: A sequence $\{x_n\}$ is monotone increasing if $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$.
(Strictly if $x_{n+1} > x_n$).

Monotone Decreasing: A sequence $\{x_n\}$ is monotone decreasing if $x_{n+1} \leq x_n, \forall n \in \mathbb{N}$.
(Strictly if $x_{n+1} < x_n$).

Theorem: If $\{x_n\}$ is monotone and bounded, then $x_n \to x$.

Nested Interval Property: $\{I_n\}$ is a sequence of closed and bounded intervals $I_n = [a_n, b_n], -\infty < a_n < b_n < \infty$ such that $\subseteq \cdots \subseteq I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_1$. So, $I_{n+1} \subseteq I_n, \forall n \in \mathbb{N}$. So, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Subsequence: Given a sequence $\{x_n\}$, consider the sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Theorem: If $\{x_n\} \subseteq \mathbb{R}$ such that $x_n \to x$, then every subsequence also converges to x.

Bolzano-Weierstrass: Every bounded sequence has a convergent subsequence.

Cauchy: A sequence $\{x_n\}$ is Cauchy if $\forall \epsilon > 0, \exists n_0(\epsilon)$ such that $|x_n - x_m| < \epsilon, \forall n, m \geq n_0$.
(Note: Convergence $\implies$ Cauchy)

Completeness of $\mathbb{R}$: $\{x_n\} \subseteq \mathbb{R}$ is Cauchy implies $x_n \to x \in \mathbb{R}$

Important Known Sequences:

Special Need to Know Sequences:

- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ $(p > 0)$
- $\lim_{n\to\infty} \sqrt[n]{p} = 1$ $(p > 0)$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \frac{n^\alpha}{p^n} = 0$ $(p > 1, \alpha \in \mathbb{R})$
- $\lim_{n\to\infty} p^n = 0$ $(|p| < 1)$
- $\lim_{n\to\infty} \frac{p^n}{n!} = 0, \forall p \in \mathbb{R}$

Limit Supremum and Limit Infimum:

Lim Sup: $\overline{\lim}_{n\to\infty} x_n = \inf_k \sup\{x_n : n \geq k\}$ $= \inf_k b_k = \lim_{k\to\infty} b_k$
Lim Inf: $\underline{\lim}_{n\to\infty} x_n = \sup_k \inf\{x_n : n \geq k\}$ $= \sup_k a_k = \lim_{k\to\infty} a_k$
Theorem: Let $\{x_n\} \subseteq \mathbb{R}$ (similar theorem holds true for lim inf)

1. $\overline{\lim}_{n\to\infty} x_n = \beta \in \mathbb{R}$ iff
   (a) (a) $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $x_n < \beta + \epsilon, \forall n \geq n_0$
   (b) (b) Given $n \in \mathbb{N}, \exists k \in \mathbb{N}$ with $k \geq n$ such that $x_k > \beta - \epsilon$.
2. $\overline{\lim}_{n\to\infty} x_n = +\infty$ iff given $M > 0$ and $n \in \mathbb{N}, \exists k \in \mathbb{N}$ such that $x_n \geq M, \forall k \geq n$.
3. $\overline{\lim}_{n\to\infty} x_n = -\infty$ iff $x_n \to -\infty$ as $n \to \infty$.

Fact: $x_n \to x$ iff $\overline{\lim} x_n = \underline{\lim} x_n$
How to: If $E = \{$subsequential limits of $x_n\}$, then

- $\overline{\lim} x_n = \sup(E)$
- $\underline{\lim} x_n = \inf(E)$

Topology on $\mathbb{R}$:

Interior Point: $E \subseteq \mathbb{R}$, $p \in E$ is an interior point if $\exists \epsilon > 0$ such that $(p-\epsilon, p+\epsilon) = N_\epsilon(p) \subseteq E$
Limit Point: $E \subseteq \mathbb{R}$, $p \in \mathbb{R}$ is a limit point of E if $\forall \epsilon > 0, \exists q \in E$ such that $q \neq p$ and $q \in N_\epsilon(p) \cap E$.
Int(E): Int(E)={all interior points of E}
E′: E′={set of all limit points of E}
Closure of E: $\overline{\text{E}}$=E ∪ E′
Open Set: O ⊆ $\mathbb{R}$ is open if Int(O)=O.
Closed Set: F ⊆ $\mathbb{R}$ is closed if F^c is open.
Theorem: For open sets...

1. For any collection $\{O_\alpha\}_{\alpha\in A}, O_\alpha \subseteq \mathbb{R}$ open $\implies \bigcup_{\alpha\in A} O_\alpha$ open.
2. $O_1, \cdots, O_n$ open $\implies \bigcap_{k=1}^n O_k$ open.

Theorem: For closed sets...

1. For $\{F_\alpha\}_{\alpha\in A}, F_\alpha \subseteq \mathbb{R}$ closed, $\forall \alpha \in A \implies \bigcap_{\alpha\in A} F_\alpha$ closed.
2. For $\{F_\alpha\}_{\alpha\in A}, F_\alpha \subseteq \mathbb{R}$ closed, $\forall \alpha \in A \implies \bigcup_{k=1}^n F_k$ closed.

Topology on $\mathbb{R}$ continued:

Theorem: $F \subseteq \mathbb{R}$ is closed $\iff$ F contains all its limit points.
Theorem: If $E \subseteq \mathbb{R}$, then

1. $\overline{E}$ is closed.
2. $E = \overline{E}$ iff E is closed.
3. $\overline{E} \subseteq F$ for every $F \subseteq \mathbb{R}$ closed such that $E \subseteq F$.

Open Cover: $E \subseteq \mathbb{R}$. $\{O_\alpha\}_{\alpha\in A}$ is an open cover (i.e. $O_\alpha \subseteq \mathbb{R}$ open) and $E \subseteq \bigcup_{\alpha\in A} O_\alpha$
Compact: $K \subseteq \mathbb{R}$ is compact if every open cover has a finite subcover. ($\exists \alpha_1, \cdots, \alpha_n \in A$ such that $K \subseteq O_{\alpha_1} \cup \cdots \cup O_{\alpha_n} = \bigcup_{k=1}^n O_{\alpha_k}$).
Theorem: Every compact subset of $\mathbb{R}$ is closed an bounded.
Heine-Borel Theorem: $[a,b] \subseteq \mathbb{R}$ is compact. $(-\infty < a, b < \infty)$
Heine-Borel-Bolzano-Weierstrass: $K \subseteq \mathbb{R}$, then TFAE:

a) K is closed and bounded.
b) K is compact.
c) Every infinite set in K has a limit point in K.

Corollary: Let $K \subseteq \mathbb{R}, K \neq \emptyset$. K is compact $\implies$ every bounded sequence has a convergent subsequence.
Cauchy-Schwartz: $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \mathbb{R}^n$. $\sum |a_k||b_k| \leq (\sum |a_k|^2)^{\frac{1}{2}} (\sum |b_k|^2)^{\frac{1}{2}}$.
Minkowski: $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \mathbb{R}^n$. $(\sum |a_k + b_k|^2)^{\frac{1}{2}} \leq (\sum |a_k|^2)^{\frac{1}{2}} + (\sum |b_k|^2)^{\frac{1}{2}}$
Höldei: $\frac{1}{p} + \frac{1}{q} = 1$.
$\sum_{k=1}^n |a_k||b_k| \leq (\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}$

Series of Real Numbers:

Theorem: If $S_n = \sum_{k=1}^n x_k$ converges, then the series $\sum_{k=1}^\infty x_k$ converges and $S = \sum_{k=1}^\infty x_k$.
Cauchy Criteria: $\sum_{k=1}^\infty x_k$ converges $\iff$ $\forall \epsilon > 0, \exists n_0(\epsilon)$ such that $|S_m - S_n| = |\sum_{k=n+1}^m x_k| < \epsilon, \forall n, m \geq n_0$.
Theorem of Convergence: If $\sum_{k=1}^\infty x_k$ converges, then $\lim_{n\to\infty} x_n = 0$.
Theorem of Divergence: If $\lim_{n\to\infty} |x_n| \neq 0$, then $\sum_{k=1}^\infty x_k$ diverges.

Convergence Tests for Series

Comparison Tests:

1. If $|x_n| \leq c_n, \forall n \geq n_0$, where n_0 is fixed, then $\sum_{k=1}^\infty c_k < \infty \implies \sum_{k=1}^\infty x_k < \infty$.
2. If $a_k \geq 0, b_k \geq 0$ and $a_k \geq b_k, \forall k \geq n_0$ (n_0 fixed), then $\sum_{k=1}^\infty b_k = +\infty \implies \sum_{k=1}^\infty a_k = +\infty$.

Limit Comparison Tests: Suppose $a_k \geq 0$ and $b_k \geq 0$. Then,

1. If $\lim_{k\to\infty} \frac{a_k}{b_k} = L, 0 < L < \infty$, then $\sum_{k=1}^\infty a_k < \infty \iff \sum_{k=1}^\infty b_k < \infty$.
2. If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^\infty b_k < \infty$, then $\sum_{k=1}^\infty a_k < \infty$.

Integral Test: Let $\{a_k\}$ be a decreasing sequence of nonnegative real numbers ($a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$). Let $f(x) : [1, \infty) \to \mathbb{R}$ and $f(x) \geq 0$ such that f is monotone decreasing and $f(k) = a_k, \forall k \in \mathbb{N}$. Then $\sum_{k=1}^\infty a_k < \infty$ iff $\int_1^\infty f(x)dx < \infty$.
Root Test: Given $\sum_{k=1}^\infty a_k$, let $\alpha = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $\alpha < 1$, then $\sum_{k=1}^\infty a_k$ converges.
2. If $\alpha > 1$, then $\sum_{k=1}^\infty a_k$ diverges.
3. If $\alpha = 1$, then the test is inconclusive.

Ratio Test: The series $\sum_{k=1}^\infty a_k$

1. converges if $\alpha = \overline{\lim}_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$.
2. diverges if $\frac{|a_{n+1}|}{a_n} \geq 1, \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Alternating Series Test: If $\{b_n\} \subseteq \mathbb{R}$ such that

1. $b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots \geq 0$
2. $\lim_{n\to\infty} b_n = 0$

then $\sum (-1)^{k+1} b_k$ converges.
Absolute Convergence: $\sum a_k$ converges absolutely if $\sum |a_k| < \infty$.
Theorem: If $\sum a_k$ converges absolutely, $\sum a_k$ converges.

Important Known Series:

	Geometric	p-Series	$n\log(n)$
	$\sum_{k=1}^{\infty} x^k$	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$
converges	$0 \leq x < 1$	$p > 1$	$p > 1$
diverges	$x \geq 1$	$p \leq 1$	$p \leq 1$

Continuous Functions:

Limit at a point: Given $L \in \mathbb{R}$, $\lim_{x\to a} f(x) = L$ if $\forall \epsilon > 0, \exists \delta(f, \epsilon, a) > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Theorem: Let f be a real-valued function defined in some neighborhood $a \in \mathbb{R}$ (including a). Then,

1. f is continuous at a. ($\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ if $|x - a| < \delta$).
2. $f(x_n) \to f(a) = L$ whenever $x_n \to a$.

Proof Outline: To show $\lim_{x\to a} f(x) = f(a)$:

1. Do scratch work to find appropriate δ by finding $|f(x) - f(a)| <$ (term involving $|x - a|$) $< \epsilon$.
2. Note that sometimes you need to chose δ to be a minimum of two things to make the inequality true. Be careful!
3. Write out proof and include scratch work.

Right Limit: $\lim_{x\to a^+} f(x) = L^+$ is the right limit if $\forall \epsilon > 0, \exists \delta(f, a, \epsilon) > 0$ such that $|f(x) - L^+| < \epsilon$ if $a < x < a + \delta$.

Left Limit: $\lim_{x\to a^-} f(x) = L^-$ is the left limit if $\forall \epsilon > 0, \exists \delta(f, a, \epsilon) > 0$ such that $|f(x) - L^-| < \epsilon$ if $a - \delta < x < a$.

Continuous at a: f is continuous at a if $f(a^+) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a^-)$

Facts: If f, g are continuous functions at a, then

- $f + g$ is continuous at a.
- fg is continuous at a.
- $\frac{1}{g}$ is continuous at a ($g(x) \neq 0$)

Composition Continuity: $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and Range$(f) \subseteq B$. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f(x) = g(f(x))$ is continuous at a.

Continuous Functions Continued:

Uniform Continuous: $f : A \subseteq \mathbb{R} \to \mathbb{R}$. f is uniformly continuous on A if $\forall \epsilon > 0, \exists \delta(f, A, \epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.
(Note: δ does NOT depend on a)

Lipschitz Continuous: $f : A \to \mathbb{R}$ is Lipschitz continuous if $\exists M > 0$ such that $|f(x) - f(y)| \leq M|x - y|, \forall x, y \in A$.

Fact: Lipschitz $\implies$ uniform $\implies$ continuous

Theorem: If $f : K \to \mathbb{R}$, $K \subseteq \mathbb{R}$ compact, and f continuous on K, then f is uniformly continuous.

Monotone Increasing: f is monotone increasing if $f(x) \leq f(y), \forall x < y$. (Strictly if $f(x) < f(y)$)

Monotone Decreasing: f is monotone decreasing if $f(x) \geq f(y), \forall x < y$. (Strictly if $f(x) < f(y)$)

Theorem: If $f : I \to \mathbb{R}$ monotone increasing on I, then $f(p^+)$ and $f(p^-)$ exists for all $p \in I$ and $\sup_{x<p} f(x) = f(p^-) \leq f(p) \leq f(p^+) = \inf_{x>p} f(x)$.

Sequences and Series of Functions:

Pointwise Limit: Let x_0 be fixed in E. Then $\{f_n(x_0)\} \subseteq \mathbb{R}$. Let $f(x_0) = n_{x_0}$. Let $\{f_n(x_0)\}$ be a sequence of functions such that $f : E \to \mathbb{R}$, then we say f_n converges pointwise on E to f if
$\forall \epsilon > 0, \exists n_0(\epsilon, x_0)$ s.t. $|f_n(x_0) - f(x_0)| < \epsilon, \forall n \geq n_0$.
So, $\lim_{n\to\infty} f_n(x_0) = f(x_0), x_0 \in E$.

Note: Interchangeability of limits, differentiation, and integration is not necessarily true when you just have pointwise continuity. You need something stronger. (Uniform continuity).

Uniform Convergence (Sequence):
a sequence $f_n : E \to \mathbb{R}$ converges uniformly on E to f if $\forall \epsilon > 0, \exists n_0(\epsilon)$ s.t. $|f_n(x) - f(x)| < \epsilon$,
$\forall n \geq n_0, \forall x \in E$.
(Note: n_0 is independent of $x \in E$)

Uniform Convergence (Series):
a series $\sum_{n=0}^{\infty} f_n(x)$; $f_n : E \to \mathbb{R}$ uniformly converges in E iff the sequence of partial sums $(S_k(x) = \sum_{n=0}^{k} f_n(x))$ are uniformly converging to $S(x)$.

Uniformly Cauchy: a sequence of functions $\{f_n(x)\}$; $f_n E \to \mathbb{R}$ is uniformly Cauchy if $\forall \epsilon < 0, \exists n_0(\epsilon)$ s.t $|f_n(x) - f_m(x)| < \epsilon, \forall n, m \geq n_0, \forall x \in E$.

Sequences and Series of Functions Continued:

Sup Norm:

- $\|f\|_\infty = \|f\|_{\text{uniform}} = \|f\|_{\sup} = \sup_{x\in K} |f(x)|$.
- $E = K$ compact $\implies \|f\|_\infty = \max_{x\in K} |f(x)|$.

Sup Norm Convergence: a sequence of functions $\{f_n\}$; $f_n : E \to \mathbb{R}$ converges in the sup norm on E if $\forall \epsilon > 0, \exists n_0(\epsilon)$ such that $\|f_n - f_m\|_\infty = \sup_{x\in E} |f_n(x) - f(x)| < \epsilon, \forall n > n_0$.

Theorem: For a sequence of functions,

Uniform Convergence
$\iff$ Uniformly Cauchy
$\iff$ Sup Norm Convergence

Theorem: $f_n : E \to \mathbb{R}$ and $f_n \in C(E)$.
If f_n converges uniformly to f on E, then $f \in C(E)$.

Proof Hint: To prove this theorem, break it up into three parts (uniformly continuous, continuous, uniformly continuous) and use the $\frac{\epsilon}{3}$ trick!

Corollary: If $\{f_n\} \subseteq (C(E), \|\cdot\|_\infty)$ is Cauchy, then f_n converges uniformly to f on $E \implies f \in C(E) \implies (C(E), \|\cdot\|_\infty)$ is complete.